Dirac's bra-ket linear algebra notation

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what it is

Dirac invented a vector notation for quantum states which is used everywhere by physicists.

$$\langle \alpha \mid \mathbf{H} \mid \beta \rangle$$

In quantum jargon, α and β are "states" (e.g. a place that a particle can be), while H is an "operator" (which does something to the particle, like bounce a photon off it). This expression represents a number, namely the probability of starting in state β, getting hit by H, and ending up at α.

But none of those details really matter for this discussion.

What matters is here this is notation for doing linear algebra. Once you have an N-dimensional basis in mind, this turns into just a $(1 \; x \, N) \cdot (N \, x \, N) \cdot (N \, x \; 1)$ matrix multiplication where α is a column vector, β is a row vector, and H is a square matrix.

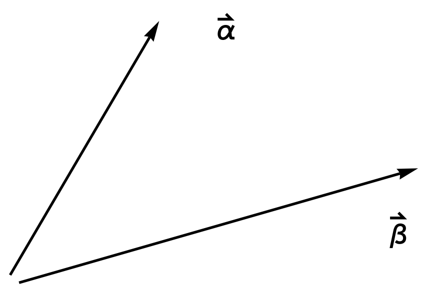
$$\begin{pmatrix} \alpha_1 & \alpha_2 & \ldots \end{pmatrix} \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \ldots \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \ldots \end{pmatrix}$$

$\langle \alpha \mid$ is called a "bra", and $\mid \beta \rangle$ is called a "ket". (This was Dirac's idea of a joke.)

This notation can be great for some sorts of linear algebra manipulations - change of basis, finding the coordinates of a vector in a given basis, using projection operators - stuff like that.

Let's start with a simple example, and convert it into this notation.

2D dot product



- **traditional notation**

In traditional notation, we'd write the dot product of the two vectors above as

$$\vec{\alpha} \cdot \vec{\beta}$$

And if we have an orthnormal basis $\hat{e}_i$ lying around, with dot products

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

then we'd write

$$\vec{\alpha} = \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2$$
$$\vec{\beta} = \beta_1 \hat{e}_1 + \beta_2 \hat{e}_2$$

You may be more used to seeing this written as $\left(\alpha_x \hat{\imath} + \alpha_y \hat{\jmath}\right)$ in the x-y plane.

Plugging in and simplifying gives

$$\begin{aligned}\vec{\alpha} \cdot \vec{\beta} &= \left(\alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2\right) \left(\beta_1 \hat{e}_1 + \beta_2 \hat{e}_2\right) \\ &= \alpha_1 \hat{e}_1 \cdot \beta_1 \hat{e}_1 + \alpha_1 \hat{e}_1 \cdot \beta_2 \hat{e}_2 + \alpha_2 \hat{e}_2 \cdot \beta_1 \hat{e}_1 + \alpha_2 \hat{e}_2 \cdot \beta_2 \hat{e}_2 \\ &= \alpha_1 \beta_1 + \alpha_2 \beta_2\end{aligned}$$

which is how you turn an abstract vector dot product into coordinates that you can actually calculate.

OK, suppose you didn't know what α_1 was, how could you write an expression for it?
I'm glad you asked. We just take the expression for $\vec{\beta}$ up above, and do a dot product with $\hat{e}_1$ on both sides :

$$\hat{e}_1 \cdot \vec{\beta} = \hat{e}_1 \cdot \left(\beta_1 \hat{e}_1 + \beta_2 \hat{e}_2\right) = \beta_1$$

- **Dirac notation**

Now it's time to put all this into Dirac's notation.

The dot product of two vectors is

$$\langle \alpha \mid \beta \rangle$$

Our basis vectors are $\mid i \rangle$ with dot products

$$\langle i \mid j \rangle = \delta_{ij}$$

But in the next step, instead of writing coordinates like β_1, instead we write explicitly $\langle 1 \mid \beta \rangle$.

Here's where it starts to get cool: this notation isn't just a name for the 1'th component of β ; it actually *is* the 1'th component of beta.

Moving right along with what we did before, we now write

$$\langle \alpha \mid = \langle \alpha \mid 1 \rangle \langle 1 \mid + \langle \alpha \mid 2 \rangle \langle 2 \mid$$
$$\langle \beta \mid = \langle \beta \mid 1 \rangle \langle 1 \mid + \langle \beta \mid 2 \rangle \langle 2 \mid$$

However, before we can now take the dot product of these two and simplify, we need to "flip" $\vec{\beta}$ from its bra form $\langle\beta\mid$ to its ket form $\mid\beta\rangle$. Technically in linear algebra jargon this is "the dual"; essentially we're just changing from a column vector to a row vector, so that the matrix multiplication works. So, flipping the second line end for end and plugging in gives

$$\begin{aligned}\langle\alpha\mid\beta\rangle &= \left(\langle\alpha\mid 1\rangle\langle 1\mid + \langle\alpha\mid 2\rangle\langle 2\mid\right)\ \left(\mid 1\rangle\langle 1\mid\beta\rangle + \mid 2\rangle\langle 2\mid\beta\rangle\right)\\ &= \langle\alpha\mid 1\rangle\langle 1\mid 1\rangle\langle 1\mid\beta\rangle + \langle\alpha\mid 1\rangle\langle 1\mid 2\rangle\langle 2\mid\beta\rangle +\\ &\quad\ \langle\alpha\mid 2\rangle\langle 1\mid\beta\rangle\langle 2\mid 1\rangle + \langle\alpha\mid 2\rangle\langle 2\mid 2\rangle\langle 2\mid\beta\rangle\\ &= \langle\alpha\mid 1\rangle\langle 1\mid\beta\rangle + \langle\alpha\mid 2\rangle\langle 2\mid\beta\rangle\end{aligned}$$

Before going any further, make sure you see that this is the same the result before : $\alpha_1\,\beta_1 + \alpha_2\,\beta_2$.

But this time, we aren't done; we can factor this part on the right to get.

$$\langle\alpha\mid\beta\rangle = \langle\alpha\mid\left(\mid 1\rangle\langle 1\mid + \mid 2\rangle\langle 2\mid\right)\mid\beta\rangle$$

or, since the part in the middle clearly is just some kind of identify multiplication

$$\mathbb{1} = \mid 1\rangle\langle 1\mid + \mid 2\rangle\langle 2\mid$$

What the heck is *that*, I can hear you say.

Well, it isn't obvious in traditional notation.

In Dirac notation, you use it all the time to "resolve" a matrix or dot product into a specific basis.
This is exactly where Dirac's notation comes into its own.

In terms of concrete things you can visualize easily, and speaking loosely and some incorrectly by writing down numbers without an explicit basis, you can think of a ket as a column vector and a bra as a row vector. So in this $\hat{e}$ basis,

$$\mid 1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad \mid 2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\langle 1\mid = \begin{pmatrix}1 & 0\end{pmatrix} \qquad \langle 2\mid = \begin{pmatrix}0 & 1\end{pmatrix}$$

and so $|1\rangle\langle 1|$ is a (2x1) times (1x2) matrix multiplication, which gives

$$\mid 1\rangle\langle 1\mid = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$\mid 2\rangle\langle 2\mid = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

and their sum is the identify matrix.

But there's also an intuitive notion of what $\mid i\rangle\langle i\mid$ means : it's a projection operator onto one component of the basis.

The vertical bars mean that each piece is ready to do a dot product in that direction. In traditional notation then, having that act on a vector might be written something like

$$\left(\cdot\ \hat{e}_1\,\hat{e}_1\ \cdot\right)\vec{\alpha} = \left(\cdot\ \hat{e}_1\,\hat{e}_1\ \cdot\right)\left(\alpha_1\,\hat{e}_1 + \alpha_2\,\hat{e}_2\right) = \hat{e}_1\,\alpha_1$$

which is the advertised projection operator : all parts of the vector except in the i'th direction have been removed.

By adding together all the projection operators, you project onto the same space, which is why for any complete orthonormal base, we get the identity

$$\mathbb{1} = |1\rangle\langle 1| + |2\rangle\langle 2| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the Operator 𝟙

By adding together all the projection operators, you project onto the same space, which is why for any complete orthonormal base, we get the identity

$$\mathbb{1} = \sum_i |i\rangle\langle i|$$

I have no idea how to write that in traditional notation. In Dirac notation, it's fundamental. To calculate anything, you stick this in all over the place to turn operators and abstract vectors into rows, columns, and matrices of numbers in a given a basic, then just turn the numerical crank.

This formula is true only for a complete, orthogonal, unit length set of basis vectors. (In quantum mechanics, that's pretty much all we care about.)

- **matrix multiplication**

I started with an expression like this.

$$\langle\alpha|\hat{H}|\beta\rangle$$

In this form we have a number, but can't calculate it. For that, we need numbers, and to get numbers, we need a basis.

Inserting the operator 𝟙 at the vertical bars resolves this into components

$$\sum_i \sum_j \langle\alpha|i\rangle\langle i|\hat{H}|j\rangle\langle j|\beta\rangle$$

which is the same as

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \ldots \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} & \ldots \\ H_{21} & H_{22} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \ldots \end{pmatrix}$$

This shows explicitly what, say H_{12} is : it's $\langle 1|\hat{H}|2\rangle$, which is an explicit recipe: let H act on basis vector 2, then dot that with basis vector 1. Or if it's more convenient to do that in another basis, you insert the operator one a few more times and do it that way.

- **change of basis**

Suppose now that we have another basis, let's call it $|\, i' \rangle$. How do we calculate the coordinates of our vector in one basis rather in terms of the other?

Well, by inserting the dirac "operator 𝟙", it's just a rote symbol manipulation game :

$$\langle i' \mid \alpha \rangle = \sum_i \langle i' \mid i \rangle \langle i \mid \alpha \rangle$$

That thing in the middle is a matrix of dot products of the basis vectors in the two bases, in all combinations.

Exercise : with an $(\hat{\imath}', \hat{\jmath}')$ basis rotated by 30 degrees clockwise from the usual (x,y) coordinates, and $\vec{\alpha} = 2\hat{\imath} + \frac{1}{2}\hat{\jmath}'$, find (α_x', α_y') by applying the last formula.

- **eigenvalues**

Eigens also fit nicely into this notation system. Say you have an operator A with eigenvalues λ_k and eigenvectors $\vec{v}_k$. Then we can use the projection operators to write that explicitly as

$$A = \sum_k \mid v_k \rangle \lambda_k \langle v_k \mid$$

Then if you want to know A's matrix representation in a specific basis, say $|\, i \rangle$, you'd calculate

$$\langle i \mid A \mid j \rangle = \sum_k \langle i \mid v_k \rangle \lambda_k \langle v_k \mid j \rangle$$

which just requires that you know the eigenvectors in that basis; everything else is plug and chug.

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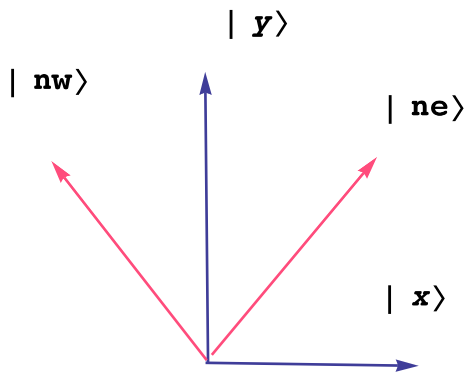
example: stretching along x=y

Let's work a problem using this notation : write down 2x2 matrix that stretches the XY plane by 2 in the x=y direction, in the standard XY basis.

Start by defining two sets of basis vectors :

$|x\rangle$ and $|y\rangle$ along the X and Y axes

$|\text{ne}\rangle$ and $|\text{nw}\rangle$ at 45 degrees to those, along the $x = y$ and $x = -y$ lines.



The dot products between these four vectors are

$$\langle \mathrm{ne} \mid \mathrm{x} \rangle = 1 \big/ \sqrt{2}$$
$$\langle \mathrm{ne} \mid \mathrm{y} \rangle = 1 \big/ \sqrt{2}$$
$$\langle \mathrm{nw} \mid \mathrm{x} \rangle = -1 \big/ \sqrt{2}$$
$$\langle \mathrm{nw} \mid \mathrm{y} \rangle = 1 \big/ \sqrt{2}$$

Now let S be the "stretch" operator that we want express as a matrix in the xy basis. From it's description, we have

$$S \mid \mathrm{ne} \rangle = 2 \mid \mathrm{ne} \rangle$$
$$S \mid \mathrm{nw} \rangle = 1 \mid \mathrm{nw} \rangle$$

or using α, β as indices that can be either ne or nw,

$$\langle \alpha \mid S \mid \beta \rangle = \begin{pmatrix} \langle \mathrm{ne} \mid S \mid \mathrm{ne} \rangle & \langle \mathrm{ne} \mid S \mid \mathrm{nw} \rangle \\ \langle \mathrm{nw} \mid S \mid \mathrm{ne} \rangle & \langle \mathrm{nw} \mid S \mid \mathrm{nw} \rangle \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Now let m, n be indices that represent either x or y. Then what we want to find is ⟨m|S |n⟩= "the stretch operator in the XY basis", by inserting the operator 𝟙 before and after S, expanding, and running the numbers. Dirac's bra-ket notation makes this sourt of calculation explicit.

$$\langle m \mid S \mid n \rangle$$
$$= \sum_\alpha \sum_\beta \langle m \mid \alpha \rangle \langle \alpha \mid S \mid \beta \rangle \langle \beta \mid n \rangle$$
$$= \begin{pmatrix} \langle x \mid \mathrm{ne} \rangle & \langle x \mid \mathrm{nw} \rangle \\ \langle y \mid \mathrm{ne} \rangle & \langle y \mid \mathrm{nw} \rangle \end{pmatrix} \begin{pmatrix} \langle \mathrm{ne} \mid S \mid \mathrm{ne} \rangle & \langle \mathrm{ne} \mid S \mid \mathrm{nw} \rangle \\ \langle \mathrm{nw} \mid S \mid \beta \rangle & \langle \mathrm{nw} \mid S \mid \mathrm{nw} \rangle \end{pmatrix} \begin{pmatrix} \langle \mathrm{ne} \mid x \rangle & \langle \mathrm{ne} \mid y \rangle \\ \langle \mathrm{nw} \mid x \rangle & \langle \mathrm{nw} \mid y \rangle \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

That last result is in fact an example of what's sometimes called "spectral decomposition", $S = U D U^*$ where D is a diagonal matrix of of the eigenvalues of S , and U is a unitary matrix of the eigenvectors of S, with $U^* = U^{-1}$, since $\mid \mathrm{ne} \rangle$ and $\mid \mathrm{nw} \rangle$ are the eigenvectors of this stretch operator, with eigenvalues 2 and 1 respectively.

the continuous limit

These same ideas can be used for some continuous variables, in particular the position x and momentum $\hbar k$ in quantum mechanics, where k with units 1/meters is the sinusoidal Fourier transform basis. The wavefunction $\Psi(x)$ can be thought of as the projection of the vector $|\Psi\rangle$ onto one particular $\langle x|$. In this "continuously infinite" vector space, the formulas for the bras and kets turn into things like

$$\langle x' \mid x\rangle = \delta(x' - x)$$

$$\langle k \mid x\rangle = \frac{1}{\sqrt{2\pi}} e^{-i k x}$$

$$\mathbb{1} = \int_{-\infty}^{+\infty} |x\rangle \langle x| \, dx = \int_{-\infty}^{+\infty} |k\rangle \, dk \, \langle k|$$

and the Forier transform turning the quantum probability wavefunction in terms of position Ψ(x) to one in momentum space $\phi(k)$ becomes just a change of basis and the application of the operator $\mathbb{1}$.

$$\phi(k) = \langle k \mid \Psi\rangle = \int_{-\infty}^{+\infty} \langle k \mid x\rangle \langle x \mid \Psi\rangle \, dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-i k x} \Psi(x) \, dx$$

But that sort of stuff is a bit beyond beyond the scope of the standard linear algebra course.

The bottom line here is that this bra-ket stuff is ubiquitous in quantum physics. While it may look a bit odd at first, it's just a linear algebra notation - one that has a nice intuition and some real power.

references

You're quickest bet to find the details is to just google "bra ket"; however, many of them spit out a lot of quantum jargon. Here are a few that discuss the notation itself.

* http://rjlipton.wordpress.com/2010/11/30/notation-and-thinking/

* http://www.conservapedia.com/Bra-ket_notation

setup and notes